

Ω-THEOREM FOR SHORT TRIGONOMETRIC SUM

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ABSTRACT. We obtain in this paper new application of the classical E.C. Titchmarsh' discrete method (1934) in the theory of the Riemann $\zeta\left(\frac{1}{2} + it\right)$ - function. Namely, we shall prove the first localized Ω -theorem for short trigonometric sum. This paper is the English version of the work of reference [4].

1. RESULT

1.1. In this paper we shall study the following short trigonometric sum

$$(1.1) \quad S(t, T, K) = \sum_{e^{-1/K} P_0 < n < P_0} \cos(t \ln n), \quad P_0 = \sqrt{\frac{T}{2\pi}}, \quad t \in [T, T + U],$$

where

$$(1.2) \quad U = T^{1/2} \psi \ln T, \quad \psi \leq K \leq T^{1/6} \ln^2 T, \quad \psi < \ln T,$$

and $\psi = \psi(T)$ stands for arbitrary slowly increasing function unbounded (from above). For example

$$\psi(T) = \ln \ln T, \quad \ln \ln \ln T, \quad \dots$$

Let

$$\{t_\nu\}$$

be the Gram-Titchmarsh sequence defined by the formula

$$\vartheta(t_\nu) = \pi \nu, \quad \nu = 1, 2, \dots$$

(see [6], pp. 221, 329) where

$$\vartheta(t) = -\frac{t}{2} \ln \pi + \operatorname{Im} \left\{ \ln \Gamma \left(\frac{1}{4} + i \frac{t}{2} \right) \right\}.$$

Next, we denote by

$$G(T, K, \psi)$$

the number of such t_ν that (see (1.1)) obey the following

$$(1.3) \quad t_\nu \in [T, T + U] \quad \wedge \quad |S(t_\nu, T, K)| > \frac{1}{2} \sqrt{\frac{P_0}{K}} = AT^{1/4} K^{-1/2}.$$

The following theorem holds true.

Theorem. There are

$$T_0(K, \psi) > 0, \quad A > 0$$

such that

$$(1.4) \quad G(T, K, \psi) > AT^{1/6} K^{-1} \psi \ln^2 T, \quad T \geq T_0(K, \psi).$$

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1.2. Let us remind the following estimate by Karatsuba (see [1], p. 89)

$$\overset{*}{S}(x) = \sum_{1 \leq n \leq x} n^{it} = \mathcal{O}(\sqrt{x}t^\epsilon), \quad 0 < x < t$$

holds true on the Lindelöf hypothesis ($0 < \epsilon$ is an arbitrary small number in this). Of course,

$$(1.5) \quad \begin{aligned} \overset{*}{S}(t, T, K) &= \sum_{e^{-1/K}P_0 \leq n \leq P_0} n^{it} = \mathcal{O}(\sqrt{P_0}T^\epsilon) = \mathcal{O}(T^{1/4+\epsilon}), \\ t &\in [T, T+U], \end{aligned}$$

and

$$|\overset{*}{S}(t, T, K)| \geq |S(t, T, K)|.$$

Remark 1. Since (see (1.1), (1.3), (1.4)) the inequality

$$|S(t, T, \ln \ln T)| > A \frac{T^{1/4}}{\sqrt{\ln \ln T}}, \quad T \rightarrow \infty$$

is fulfilled for arbitrary big t , then the Karatsuba's estimate (1.5) is an almost exact estimate.

1.3. With regard to connection between short trigonometric sum and the theory of the Riemann zeta function see our paper [3].

2. MAIN LEMMAS AND PROOF OF THEOREM

Let

$$(2.1) \quad w(t, T, K) = \sum_{e^{-1/K}P_0 < n < P_0} \frac{1}{\sqrt{n}} \cos(t \ln n),$$

$$(2.2) \quad w_1(t, T, K) = \sum_{e^{-1/K}P_0 < n < P_0} \left(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{P_0}} \right) \cos(t \ln n).$$

The following lemmas hold true.

Lemma A.

$$(2.3) \quad \sum_{T \leq t_\nu \leq T+U} w^2(t_\nu, T, K) = \frac{1}{4\pi} U K^{-1} \ln \frac{T}{2\pi} + \mathcal{O}(K^{-1} \sqrt{T} \ln^2 T).$$

Lemma B.

$$(2.4) \quad \sum_{T \leq t_\nu \leq T+U} w_1^2(t_\nu, T, K) = \frac{1}{48\pi} U K^{-3} \ln \frac{T}{2\pi} + \mathcal{O}(K^{-3} \sqrt{T} \ln^2 T).$$

Remark 2. Of course, the formulae (2.3), (2.4) are asymptotic ones (see (1.2)).

Now we use the main lemmas A and B for completion of the Theorem. Since (see [2], (23))

$$(2.5) \quad Q_0 = \sum_{T \leq t_\nu \leq T+U} 1 = \frac{1}{2\pi} U \ln \frac{T}{2\pi} + \mathcal{O}\left(\frac{U^2}{T}\right),$$

then we obtain (see (2.3), (2.4)) that

$$(2.6) \quad \frac{1}{Q_0} \sum_{T \leq t_\nu \leq T+U} w^2(t_\nu, T, K) \sim \frac{1}{2K}, \quad T \rightarrow \infty,$$

$$(2.7) \quad \frac{1}{Q_0} \sum_{T \leq t_\nu \leq T+U} w_1^2(t_\nu, T, K) \sim \frac{1}{24K^3},$$

$$(2.8) \quad \frac{1}{Q_0} \sum_{T \leq t_\nu \leq T+U} w \cdot w_1 = \mathcal{O}\left(\frac{1}{K^2}\right),$$

(we used the Schwarz inequality in (2.8)). Next, we have (see (1.1), (2.1), (2.2)) that

$$(2.9) \quad w(t_\nu, T, K) = \frac{1}{\sqrt{P_0}} S(t_\nu, T, K) + w_1(t_\nu, T, K).$$

Consequently we obtain (see (2.6) – (2.9)) the following

Formula.

$$(2.10) \quad \frac{1}{Q_0} \sum_{T \leq t_\nu \leq T+U} S^2 \sim \frac{P_0}{2K}, \quad T \rightarrow \infty.$$

Next, we denote by Q_1 the number of such values

$$t_\nu \in [T, T+U],$$

that fulfill the inequality (see (1.3))

$$(2.11) \quad |S| > \frac{1}{2} \sqrt{\frac{P_0}{K}}; \quad Q_1 = G(T, K, \psi),$$

and

$$Q_0 - Q_1 = Q_2$$

(see (2.5)). Since (see (1.1) and [6], p. 92)

$$|S(t, T, K)| < A\sqrt{P_0}T^{1/6}, \quad t \in [T, T+U],$$

then we have (see (2.10), (2.11)) that

$$\frac{1}{3K} < AT^{1/3} \frac{Q_1}{Q_0} + \frac{1}{4K} \frac{Q_2}{Q_0} < AT^{1/3} \frac{Q_1}{Q_0} + \frac{1}{4K},$$

i. e.

$$(2.12) \quad AQ_1 > \frac{1}{12} Q_0 T^{-1/3} K^{-1}.$$

Consequently, we obtain (see (1.2), (2.5), (2.11), (2.12)) the following estimate

$$Q_1 = G > AT^{1/6} K^{-1} \psi \ln^2 T$$

that is required result (1.4).

3. LEMMA 1

Let

$$(3.1) \quad w_2 = \sum_{e^{-1/K} P_0 < n < m < P_0} \sum \frac{1}{\sqrt{nm}} \cos \left(t_\nu \ln \frac{n}{m} \right).$$

The following lemma holds true.

Lemma 1.

$$(3.2) \quad \sum_{T \leq t_\nu \leq T+U} w_2 = \mathcal{O}(K^{-1} \sqrt{T} \ln^2 T).$$

Proof. The following inner sum (comp. [5], p. 102; $t_{\nu+1} \rightarrow t_\nu$)

$$(3.3) \quad w_{21} = \sum_{T \leq T_\nu \leq T+U} \cos \{2\pi \psi_1(\nu)\},$$

where

$$\psi_1(\nu) = \frac{1}{2\pi} t_\nu \ln \frac{n}{m}$$

applies to our sum (3.2). Now we obtain by method [5], pp. 102-103 the following estimate

$$(3.4) \quad w_{21} = \mathcal{O} \left(\frac{\ln T}{\ln \frac{n}{m}} \right).$$

Since (see (1.2))

$$e^{-1/K} > 1 - \frac{1}{K} \geq 1 - \frac{1}{\psi} > \frac{1}{2},$$

then

$$2m > 2e^{-1/K} P_0 > P_0, \quad m \in (e^{-1/K} P_0, P_0),$$

i. e. in our case (see (3.1)) we have that

$$2m > n.$$

Consequently, the method [6], p. 116, $\sigma = \frac{1}{2}$, $m = n - r$ gives the estimate

$$(3.5) \quad \begin{aligned} & \sum_{e^{-1/K} P_0 < n < m < P_0} \sum \frac{1}{\sqrt{mn} \ln \frac{n}{m}} < \\ & < A \sum_{e^{-1/K} P_0 < n < P_0} \sum_{r \leq n/2} \frac{1}{r} < AK^{-1} P_0 \ln P_0 < AK^{-1} \sqrt{T} \ln T, \end{aligned}$$

where

$$(3.6) \quad \sum_{e^{-1/K} P_0 < n < P_0} 1 \sim \frac{P_0}{K}.$$

Now, required result (3.2) follows from (3.1), (3.3) – (3.5). □

4. LEMMA 2

Let

$$(4.1) \quad w_3 = \sum_{e^{-1/K} P_0 < m < n < P_0} \sum \frac{1}{\sqrt{mn}} \cos\{t_\nu \ln(mn)\}.$$

The following lemma holds true.

Lemma 2.

$$(4.2) \quad \sum_{T \leq t_\nu \leq T+U} w_3 = \mathcal{O}(K^{-1} \sqrt{T} \ln^2 T).$$

Proof. The following inner sum (comp. [5], p. 103; $t_{\nu+1} \rightarrow t_\nu$)

$$w_{31} = \sum_{T \leq t_\nu \leq T+U} \cos\{2\pi\chi(\nu)\},$$

where

$$\chi(\nu) = \frac{1}{2\pi} t_\nu \ln(nm)$$

applies to our sum (4.2). Next, the method [5], pp. 103-104 gives us that

$$(4.3) \quad \begin{aligned} w_{31} &= \int_{\chi'(x) < 1/2} \cos\{2\pi\chi(x)\} dx + \\ &+ \int_{\chi'(x) > 1/2} \cos[2\pi\{\chi(x) - x\}] dx + \mathcal{O}(1) = J_1 + J_2 + \mathcal{O}(1), \end{aligned}$$

where

$$J_1 = \mathcal{O}\left(\frac{\ln T}{\ln n}\right) = \mathcal{O}(1), \quad n \in (e^{-1/K} P_0, P_0),$$

and ($m < n < 2m$, $n = m + r$)

$$J_2 = \mathcal{O}\left(\frac{m \ln(m+1)}{r}\right).$$

Now, the term J_1 contributes to the sum (4.2), (comp. (3.6)) as

$$(4.4) \quad \begin{aligned} &\mathcal{O}\left(\sum_{e^{-1/K} P_0 < m < n < P_0} \sum \frac{1}{\sqrt{mn}}\right) = \\ &= \mathcal{O}\left(\frac{1}{P_0} \sum_{e^{-1/K} P_0 < m < n < P_0} \sum 1\right) = \\ &= \mathcal{O}\left(\frac{1}{P_0} \frac{P_0^2}{K^2}\right) = \mathcal{O}(K^{-2} \sqrt{T}), \end{aligned}$$

and the same contribution corresponds to the term $\mathcal{O}(1)$ in (4.3), while the contribution of the term J_2 is

$$(4.5) \quad \begin{aligned} &\mathcal{O}\left(\sum_{e^{-1/K} P_0 < m < P_0} \frac{1}{\sqrt{m}} \sum_{r=1}^m \frac{1}{\sqrt{m}} \frac{m \ln(m+1)}{r}\right) = \\ &= \mathcal{O}\left(\frac{P_0}{K} \ln^2 P_0\right) = \mathcal{O}(K^{-1} \sqrt{T} \ln^2 T). \end{aligned}$$

Now, the required result (4.2) follows from (4.1), (4.4), (4.5). □

5. LEMMA 3

Let

$$(5.1) \quad w_4 = \sum_{e^{-1/K} P_0 < n < P_0} \frac{1}{n}.$$

The following lemma holds true.

Lemma 3.

$$(5.2) \quad \sum_{T \leq t_\nu \leq T+U} w_4 = \mathcal{O}(K^{-1} \sqrt{T} \ln^2 T).$$

Proof. The following inner sum

$$w_{41} = \sum_{T \leq t_\nu \leq T+U} \cos\{2\pi\chi_1(\nu)\},$$

where

$$\chi_1(\nu) = \frac{1}{\pi} t_\nu \ln n.$$

applies to our sum (5.2). Since (comp. [5], p. 103)

$$\chi'_1(\nu) = \frac{\ln n}{\vartheta'(t_\nu)},$$

next, (comp. [5], p. 100)

$$\begin{aligned} \vartheta'(t_\nu) &= \frac{1}{2} \ln \frac{t_\nu}{2\pi} + \mathcal{O}\left(\frac{1}{t_\nu}\right) = \frac{1}{2} \ln \frac{T}{2\pi} + \mathcal{O}\left(\frac{U}{T}\right) + \mathcal{O}\left(\frac{1}{T}\right) \sim \\ &\sim \ln P_0, \end{aligned}$$

and

$$\ln P_0 - \frac{1}{K} < \ln n < \ln P_0, \quad n \in (e^{-1/K} P_0, P_0),$$

then

$$\chi'_1(\nu) \sim 1.$$

Since, for example,

$$\frac{1}{2} < \chi'_1(\nu) < \frac{3}{2},$$

then we have (comp. [5], p. 104) that

$$w_{41} = \int \cos[2\pi\{\chi_1(x) - x\}] dx + \mathcal{O}(1) = J_3 + \mathcal{O}(1).$$

Now, (comp. [5], p. 104)

$$\chi''_1(\nu) < -A \frac{\ln n}{T \ln^3 T} < -\frac{B}{T \ln^2 T}, \quad n \in (e^{-1/K} P_0, P_0),$$

and

$$J_3 = \mathcal{O}(\sqrt{T} \ln T), \quad w_{41} = \mathcal{O}(\sqrt{T} \ln T).$$

Consequently, we get the required result (5.2)

$$\sum_{T \leq t_\nu \leq T+U} w_4 = \mathcal{O}\left(\sqrt{T} \ln T \sum_{e^{-1/K} P_0 < n < P_0} \frac{1}{n}\right) = \mathcal{O}(K^{-1} \sqrt{T} \ln T),$$

where

$$\sum_{e^{-1/K}P_0 < n < P_0} \frac{1}{n} \sim \frac{1}{K}$$

by the well-known Euler's formula

$$\sum_{1 \leq n < x} \frac{1}{n} = \ln x + c + \mathcal{O}\left(\frac{1}{x}\right),$$

where c is the Euler's constant. □

6. LEMMAS A AND B

6.1. Proof of Lemma A. First of all, we have (see (2.1)) that

$$\begin{aligned} w^2(t_\nu, T, K) &= \\ &= \sum_{e^{-1/K}P_0 < m, n < P_0} \frac{1}{\sqrt{mn}} \cos(t_\nu \ln m) \cos(t_\nu \ln n) = \\ (6.1) \quad &= \frac{1}{2} \sum_n \frac{1}{n} + \sum_{m < n} \sum \frac{1}{\sqrt{mn}} \cos\left(t_\nu \ln \frac{n}{m}\right) + \\ &+ \sum_{m < n} \sum \frac{1}{\sqrt{mn}} \cos\{t_\nu \ln(mn)\} + \frac{1}{2} \sum_n \frac{1}{n} \cos(2t_\nu \ln n) = \\ &= \frac{1}{2K} + \mathcal{O}\left(\frac{1}{\sqrt{T}}\right) + w_2 + w_3 + w_4, \end{aligned}$$

(see (5.3), (3.1), (4.1), (5.1)). Consequently, we obtain the required result (2.3) from (6.1) by (2.5), (3.2), (4.2), (5.2).

6.2. Proof of Lemma B. First of all we have (see (2.2)) that

$$w_1(t_\nu, T, K) = \sum_{e^{-1/K}P_0 < n < P_0} \frac{\alpha(n)}{\sqrt{n}} \cos(t_\nu \ln n),$$

where

$$\alpha(n) = 1 - \sqrt{\frac{n}{P_0}}.$$

Of course, $\alpha(n)$ is decreasing and

$$(6.2) \quad 0 < \alpha(n) < \frac{1}{K}, \quad n \in (e^{-1/K}P_0, P_0).$$

Next, (comp. (6.1))

$$\begin{aligned} w_1^2(t_\nu, T, K) &= \\ &= \frac{1}{2} \sum_n \frac{\alpha^2(n)}{n} + \sum_{m < n} \sum \frac{\alpha(m)\alpha(n)}{\sqrt{mn}} \cos\left(t_\nu \ln \frac{n}{m}\right) + \\ (6.3) \quad &+ \sum_{m < n} \sum \frac{\alpha(m)\alpha(n)}{\sqrt{mn}} \cos(t_\nu \ln(mn)) + \frac{1}{2} \sum_n \frac{\alpha^2(n)}{n} \cos(2t_\nu \ln n) = \\ &= \frac{1}{2} \bar{w}_1 + \bar{w}_2 + \bar{w}_3 + \frac{1}{2} \bar{w}_4. \end{aligned}$$

Since (see (6.2))

$$\alpha(m)\alpha(n) < K^{-2}$$

then we obtain by a similar way as in the case of the estimates (3.2), (4.2) and (5.2) that

$$(6.4) \quad \sum_{T \leq t_\nu \leq T+U} \left\{ \bar{w}_2 + \bar{w}_3 + \frac{1}{2} \bar{w}_4 \right\} = \mathcal{O}(K^{-3} \sqrt{T} \ln^2 T).$$

In the case of the sum

$$(6.5) \quad \frac{1}{2} \sum_{T \leq t_\nu \leq T+U} \bar{w}_1$$

we use the following summation formula (see [6], p. 13)

$$\begin{aligned} \sum_{a \leq n < b} \varphi(n) &= \int_a^b \varphi(x) dx + \int_a^b \left(x - [x] - \frac{1}{2} \right) \varphi'(x) dx + \\ &+ \left(a - [a] - \frac{1}{2} \right) \varphi(a) - \left(b - [b] - \frac{1}{2} \right) \varphi(b) \end{aligned}$$

in the case

$$a = e^{-1/K} P_0, b = P_0, \varphi(x) = \frac{\alpha^2(x)}{x} = \frac{1}{x} - \frac{2}{\sqrt{P_0 x}} + \frac{1}{P_0}.$$

Hence

$$\begin{aligned} \int_{e^{-1/K} P_0}^{P_0} \frac{\alpha^2(x)}{x} dx &= \frac{1}{K} - 4 \left(1 - e^{-1/(2K)} \right) + 1 - e^{-1/K} = \\ &= \frac{1}{12K^3} + \mathcal{O}(K^{-4}), \end{aligned}$$

and

$$\varphi'(x) = \mathcal{O}(P_0^{-2}), \quad \varphi(e^{-1/K} P_0) = \mathcal{O}\left(\frac{1}{x^2 P_0}\right), \quad \varphi(P_0) = 0.$$

Consequently, we have (see (6.5))

$$\frac{1}{2} \bar{w}_1 = \frac{1}{24K^3} + \mathcal{O}\left(\frac{1}{K^4}\right),$$

and (see (1.2), (2.5))

$$\begin{aligned} (6.6) \quad \frac{1}{2} \sum_{T \leq t_\nu \leq T+U} \bar{w}_1 &= \frac{1}{48\pi} U K^{-3} \ln \frac{T}{2\pi} + \mathcal{O}\left(\frac{U \ln T}{K^4}\right) + \mathcal{O}\left(\frac{U^2}{K^3 T}\right) = \\ &= \frac{1}{48\pi} U K^{-3} \ln \frac{T}{2\pi} + \mathcal{O}(K^{-3} \sqrt{T} \ln T). \end{aligned}$$

Finally, we obtain the required result (2.4) from (6.3) by (6.4) – (6.6).

REFERENCES

- [1] A. A. Karatsuba, ‘*Basic analytic number theory*’, Moscow, (1975), (in Russian).
- [2] J. Moser, ‘On one theorem of Hardy-Littlewood in the theory of the Riemann zeta-function’, Acta Arith. 31, (1976), 45-51; 40 (1981), 97-107, (in Russian).
- [3] J. Moser, ‘A new estimate of short trigonometric sum’, Acta Arith., 40 (1980), 357-367, (in Russian).
- [4] J. Moser, ‘ Ω -theorem for short trigonometric sum’, Acta Arith. 42 (1983), 153-161, (in Russian).
- [5] E. C. Titchmarsh, ‘On van der Corput’s method and the zeta-function of Riemann, (IV)’, Quart. J. Math. 5, (1934), 98-105.
- [6] E. C. Titchmarsh, ‘*The theory of the Riemann zeta-function*’, Clarendon Press, Oxford, 1951.

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